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COMMENTS ON STIGLITZ'S RE-EXAMINATION OF THE MODIGLIANI-MILLER THEOREM

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Summary:

The purpose of this paper is to clarify two misstatements in Joseph Stiglitz's landmark work, "A Re-Examination of the Modigliani-Miller Theorem." Although these revisions are essential for the original paper to stand alone, they do not alter the thrust of Stiglitz's thesis or his conclusions.

Comments on Stiglitz's Re-Examination
of the Modigliani-Miller Theorem

The purpose of this paper is to isolate two apparent misstatements in Joseph Stiglitz's landmark paper, "A Re-Examination of the Modigliani-Miller Theorem." Its intent is not to discredit but clarify. Insertion of the suggested changes does <u>not</u> alter Stiglitz's conclusions. However, these inclusions are necessary for the paper to stand by itself.

Since the appearance of Modigliani and Miller's (M&M) classic paper, legitimate criticisms of their remarkable conclusions have focused on the validity of their assumptions. Stiglitz enumerates five limitations of the M&M proof. These are:

"1. It depend(s) on the existence of risk classes.

2. The use of risk classes seem(s) to imply objective rather than subjective probability distributions over the possible outcomes.

3. It (is) based on partial equilibrium rather than general equilibrium analysis.

4. It (is) not clear whether the theorem held only for competitive markets.

5. Except under special circumstances, it (is) not clear how the possibility of firm bankruptcy affected the validity of the theorem." [p. 784]

Through a general equilibrium, state preference model, Stiglitz demonstrates that M&M's Proposition I[1] ". . . holds under much more general conditions than those assumed in their original study. The validity of the theorem does not depend on the existence of risk classes, on compet-

itiveness of the capital market, or on the agreement of individuals about the probability distribution of outcomes." [Stiglitz, p. 784]

Two of M&M's assumptions are critical. These require that all market participants (firms and individuals) can borrow at the same market rate of interest, and that there is no risk of bankruptcy. Under a more stringent set of conditions, the M&M results will still be valid even though the existence of limitations on individual borrowing and/or the possibility of bankruptcy is present. In detailing these more stringent conditions, Stiglitz has erred.

Limitations on Individual Borrowing

Stiglitz correctly refutes the objection that individuals cannot borrow at the same rate of interest as firms by emphasizing that individuals need not "actually borrow from the market but only change their holdings of bonds." Therefore he continues, "a problem can arise then only if an individual has no bonds in his portfolio." [p. 787] In equation (9) Stiglitz stipulates an inequality that constrains all individuals to be lenders.

$$(9) \quad \sum_i \alpha_i^j B_i \geq w^j - \sum_i \alpha_i^j V_i \qquad \text{(for all } j\text{)}$$

where α_i^j = the percentage of the i^{th} firm's equity owned by the j^{th} individual, E_i^j/E_i [2],

V_i = the value of firm i,

E_i = the value of the i^{th} firm's equity,

B_i = the value of the i^{th} firm's bonds, and

w^j = the value of the j^{th} individual's wealth.

Clearly then, $V_i \equiv E_i + B_i$.

If we assume an environment of 3 individuals (j = a, b, c) and two firms (i = 1, 2) with market values:

$$V_1 = E_1 + B_1,$$

$$V_2 = E_2 + B_2,$$

where E_i, $B_i > 0$ (i = 1, 2),

constraint (9) yields:

for j = a $\qquad w^a - \alpha_1^a E_1 - \alpha_2^a E_2 \leq 2(\alpha_1^a B_1 + \alpha_2^a B_2),$

for j = b $\qquad w^b - \alpha_1^b E_1 - \alpha_2^b E_2 \leq 2(\alpha_1^b B_1 + \alpha_2^b B_2),$ and

for j = c $\qquad w^c - \alpha_1^c E_1 - \alpha_w^c E_2 \leq 2(\alpha_1^c B_1 + \alpha_2^c B_2).$

These inequalities make very little economic sense. Moreover, it is not intuitively clear why this set describes the necessary and sufficient conditions for all individuals to be lenders. Equation (9') is a direct approach that requires all individuals to be lenders.

$$(9') \quad w^j - \sum_i \alpha_i^j E_i > 0 \qquad \text{(for all j)}$$

This inequality forces the wealth of each individual, w^j, minus the sum of his equity holdings in each firm, to be positive. Since all wealth is defined in terms of E_i^j or B^j, the positive difference defined in (9') must be B^j. Thus if (9') holds for every (j^{th}) individual, everyone will own bonds (i.e. will be a lender). We can recast (9') in terms of V_i rather than E_i and offset the additional B_i terms on righthand side of (9') as follows:

$$(9'') \qquad w^j - \sum_i \alpha_i^j V_i > - \sum_i \alpha_i^j B_i,$$

which expands to

$$w^j - \sum_i \alpha_i^j E_i - \sum_i \alpha_i^j B_i > - \sum_i \alpha_i^j B_i.$$

Because the B_i terms cancel, (9') and (9'') are identical. Interestingly, (9'') may be transposed to give (9''').

$$(9''') \qquad \sum_i \alpha_i^j B_i > - [w^j - \sum_i \alpha_i^j V_i]$$

$$(9) \qquad \sum_i \alpha_i^j B_i \geq [w^j - \sum_i \alpha_i^j V_i]$$

Equations (9''') and (9) are strikingly similar. It is possible that the difference results from a typographical error.

To check the suitability of the two formulations we can extend our simple financial environment consisting of two firms (i = 1, 2), three individuals (j = a, b, c), and market values:

$V_1 = E_1 + B_1$ and $V_2 = E_2 + B_2$,

where E_i, $B_i > 0$ (i = 1, 2). Also assume that:

$$\alpha_1^a = 1/2, \quad \alpha_2^a = 1/2,$$
$$\alpha_1^b = 1/2, \quad \alpha_2^b = 1/2,$$
$$\alpha_1^c = 0, \quad \alpha_2^c = 0.$$

If individual a owns 1/2 of B_1 and individual b owns the remaining 1/2 B_1, this implies:

$$w^a = 1/2\ E_1 + 1/2\ E_2 + 1/2\ B_1,$$
$$w^b = 1/2\ E_1 + 1/2\ E_2 + 1/2\ B_1, \quad \text{and}$$
$$w^c = B_2.$$

Plugging these values into Stiglitz's equation (9) infers:

$$1/2\ B_1 + B_2 \geq 0 \quad \text{for } j = a \quad \text{(correct),}$$ [3]

$$1/2\ B_1 + B_2 \geq 0 \quad \text{for } j = b \quad \text{(correct), and}$$

$$B_2 \leq 0 \quad \text{for } j = c \quad \text{(incorrect).}$$

However, if these values are applied to equation (9'''), they yield:

$$1/2\ B_1 > 0 \quad \text{for } j = a \quad \text{(correct),}$$

$$1/2\ B_1 > 0 \quad \text{for } j = b \quad \text{(correct), and}$$

$$B_2 > 0 \quad \text{for } j = c \quad \text{(correct).}$$

On the other hand, if only individual c own bonds, while individuals a and b own only equity (in the previous proportions), the following w^j would be appropriate:

$$w^a = 1/2\ E_1 + 1/2\ E_2,$$

$$w^b = 1/2\ E_1 + 1/2\ E_2, \quad \text{and}$$

$$w^c = B_1 + B_2.$$

Stiglitz's equation (9) infers:

$$B_1 + B_2 \geq 0 \quad \text{for } j = a \quad \text{(incorrect),}$$

$$B_1 + B_2 \geq 0 \quad \text{for } j = b \quad \text{(incorrect), and}$$

$$B_1 + B_2 \leq 0 \quad \text{for } j = c \quad \text{(incorrect),}$$

while (9''') gives:

$$0 > 0 \quad \text{for } j = a \quad \text{(correct),}$$

$$0 > 0 \quad \text{for } j = b \quad \text{(correct), and}$$

$$B_1 + B_2 > 0 \quad \text{for } j = c \quad \text{(correct).}$$

Thus it appears (9''') is the appropriate constraining condition to insure that all individuals hold some bonds in a portion of their portfolio.

Mean Variance Analysis and the Separation Theorem in the Presence of Bankruptcy

In section III, Stiglitz establishes that even if a firm issues enough bonds to go bankrupt, as long as all individuals agree on the probability distribution of returns for each firm, the value of that "risky" firm is independent of its debt equity ratio.

Most treatments of mean-variance analysis focus on rates of return. Indeed, those familiar with the Sharpe-Lintner capital asset pricing model (CAPM) should recognize Stiglitz's assumption as a restatement of the homogeneous expectations or "idealized" uncertainty requirement. Fama details several alternative versions of the CAPM.[4]

$$\text{(1)} \qquad \varepsilon(r_i) = r^* + \beta_i[\varepsilon(r_m) - r^*],$$

$$\text{(1')} \qquad = r^* + \frac{[\varepsilon(r_m) - r^*]\ \mathrm{cov}(r_i, r_m)}{\sigma^2(r_m)},$$

$$\text{(1'')} \qquad = r^* + \Lambda\ \mathrm{cov}(r_i, r_m),$$

where r_i = the rate of return on the ith risky asset (or portfolio),

r^* = the rate of return on a risk-free asset (i.e. the risk-free rate),

r_m = the rate of return on the market portfolio of risky assets,

$\text{cov}(r_i, r_m)$ = the covariance of return on the i^{th} asset with the market return,

$\sigma^2(r_m)$ = the variance of the market return,

$\beta_i = \text{cov}(r_i, r_m)/\sigma^2(r_m)$,

$\Lambda = \dfrac{\varepsilon(r_m) - r^*}{\sigma^2(r_m)}$ = the market price of risk, and

ε = the expectations operator.

Instead of focusing on rates of return, Stiglitz (referring to Lintner) places his emphasis on stock and flow variables (i.e. market values and "operating" dollar returns or operating income.[5] Although one set of variables (rates of return) is the mirror image of the other (value and dollar returns) and vice versa, the transition between variables is not always smooth.[6] Nevertheless, it should be evident that if Stiglitz's presentation is correct, it should collapse into the CAPM.

We will reconstruct Stiglitz's equations (10) through (12), based upon Lintner's equations (29) through (29'') because they served as Stiglitz's point of embarkation.

$$(29) \qquad V_i = (\overline{R}_i - W_i)/r^*,$$

$$(29') \qquad = [\overline{R}_i - (\lambda/T) \sum_j \check{R}_{ij}]/r^*,$$

$$(29'') \qquad = [\overline{R}_i - \frac{\sum_i (\overline{R}_i - r^* V_i)}{\sum\sum_{ij} \check{R}_{ij}} \sum_j \check{R}_{ij}]/r^*,$$

where V_i = aggregate market value of firm i,

$\overline{R}_i$ = the expected level of operating income for firm i,

$\overset{v}{R}_{ij}$ = the covariance of the operating incomes of firms i and j, and

r^* = the risk-free rate.

Stiglitz's equation (10) is a restatement of (29'); however, a more precise version is (10').

$$(10) \qquad E_i + B_i = \left\{\overline{X} - k \sum_{j=i}^{n} \varepsilon(X_1 - \overline{X}_i)(X_j - \overline{X}_j)\right\} / r^*$$

$$(10') \qquad V_i = E_i + B_i = \left\{\overline{X}_i - k \sum_{j=1}^{n} \varepsilon[(X_i - \overline{X}_i)(X_j - \overline{X}_j)]\right\} / r^*,$$

where E_i, B_i, B_i, ε and r^* are as previously defined and

$\overline{X}_i$ = the expected level of operating income for firm i.

Since $X_i \equiv$ Lintner's R_i, we can rewrite (10') as follows:

$$(10'') \qquad V_i = \left\{\overline{X}_i - k \sum_{j=1}^{n} \overset{v}{X}_{ij}\right\} / r^*.$$

Comparing (10'') with (29') it should be clear that $k \equiv \lambda/T$.

Stiglitz defines k as:

$$k = r^* \left[\sum_i (X_i - \overline{X}_i) \Big/ \sum_{ij}\sum \varepsilon(X_i - \overline{X}_i)(X_j - \overline{X}_j)\right].$$

However, since $\sum_i (X_i - \overline{X}_i)$ would equal zero, k would equal zero.[7] It should be evident that this was not Stiglitz's intent. If this were true, the market value of every firm would equal the value of its gross dollar returns capitalized at the risk-free rate. By writing Stiglitz's variables into Lintner's structure, we can derive a new k.

$$k = \frac{\sum_i (\overline{X}_i - r^* V_i)}{\sum_{ij}\sum \overset{v}{X}_{ij}} \quad ,$$

$$= \frac{\sum_i (\overline{X}_i - r^*V_i)}{\sum\sum_{ij} \varepsilon[(X_i - \overline{X}_i)(X_j - \overline{X}_j)]} ,$$

$$= \frac{\text{the excess dollar returns (flows) of the market (excess above the risk-free rate times the aggregate market value of all firms comprising the market)}}{\text{the variance of the dollar market return}} .$$

Substituting this new k into (10') yields a close surrogate for Lintner's original equation (29).

$$\text{(2)} \qquad V_i = (\overline{X}_i - \Omega_i) / r^*,$$

$$\text{where } \Omega_i = \frac{\sum_i (\overline{X}_i - r^*V_i) \sum_j \varepsilon[(X_i - \overline{X}_i)(X_j - \overline{X}_j)]}{\sum\sum_{ij} \varepsilon\, [(X_i - \overline{X}_i)(X_j - \overline{X}_j)]} .$$

Multiplying Ω_i by $1/\sum_i V_i \sum_j V_j \big/ 1/\sum_i V_i \sum_j V_j$ does not change its value; however, it transforms dollar flows (X_i and $\overline{X}_i$) into rates of return (r_i and $\overline{r}_i$). This procedure yields:

$$\Omega_i = \frac{[\varepsilon(r_m) - r^*] \sum_j \varepsilon[(X_i - \overline{X}_i)(r_j - \overline{r}_j)]}{\sum\sum_{ij} \varepsilon\, [(r_i - \overline{r}_i)(r_j - \overline{r}_j)]} .$$

A closer inspection of Ω_i shows that

$$\sum_j \varepsilon\, [(X_i - \overline{X}_i)(r_j - \overline{r}_j)] = \mathrm{cov}(X_i, r_m),$$

and

$$\sum\sum_{ij} \varepsilon\, [(r_i - \overline{r}_i)(r_j - \overline{r}_j)] = \mathrm{cov}(r_m, r_m) = \sigma^2(r_m).$$

Inserting these terms into (2) gives

$$(2') \quad V_i = \left\{ \bar{X}_i - \frac{[\varepsilon(r_m) - r^*]\mathrm{cov}(X_i, r_m)}{\sigma^2(r_m)} \right\} / \ r^* \quad .$$

By dividing both sides of (2') by V_i and cross-multiplying, we obtain

$$r^* = \varepsilon(r_i) - \frac{[\varepsilon(r_m) - r^*]\mathrm{cov}(r_i, r_m)}{\sigma^2(r_m)} \quad ,$$

<u>or</u> $\quad \varepsilon(r_i) = r^* + \beta_i[\varepsilon(r_m) - r^*],$

where $\quad \beta_i = \dfrac{\mathrm{cov}(r_i, r_m)}{\sigma^2(r_m)} \quad ,$

which is of course identical to (1). Thus, the reformulation of k must be correct.

Conclusions

This paper has isolated two revisions required for exactness and rigor. The first is a reformulation of Stiglitz's equation (9). The second restates a variable, k, utilized in equations (10) through (12). Although the corrections are essential for the original paper to stand alone, they do <u>not</u> alter the thrust of Stiglitz's thesis or his conclusions.

FOOTNOTES

[1]Proposition I states ". . . the market value of any firm is independent of its capital structure and is given by capitalizing its expected return at the rate ρ_k appropriate to its risk class" . . . "the average cost of capital to any firm is completely independent of its capital structure and is equal to the capitalization rate of a pure equity stream of its class." [Modigliani & Miller, p. 268]

[2]The superscript refers to individuals; the subscript pertains to firms.

[3]This remark refers to the constraint's suitability and accuracy in describing the given financial environment. For example, if each individual in our hypothetical environment owns bonds, the constraint must describe a feasible condition for every individual. On the other hand, if the environment has an individual who owns no bonds, the constraint must describe an infeasible condition for at least one individual.

[4]Admittedly a precise statement of equations (1-1'') would include complementary random error terms. Fortunately for our purposes, the assumptions of the model (see Fama, pp. 37-40) obviate this requirement.

[5]Conventionally, "operating income" is income before financial charges. Stiglitz uses the term gross returns, i.e. the dollar return before paying bondholders but after paying all non-capital factors of production.

[6]As a general rule rates of return are represented as lowercase letters; values and dollar flows are uppercase letters.

[7]This, of course, assumes that X_i is a normally distributed random variable.

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